Right Turn on Red at Signalised Intersections
A Stochastic Evaluation Approach

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1 Context

In design and dimensioning of an intersection the criteria safety, traffic quality, as well as townscape have priority. It is the task of the planning engineer to find the optimum solution which fulfils these criteria within a variety of possible solutions and combinations.

The dimensioning, in particular, is covered by extensive standards for the various node types and right of way regulations. The applicable literature, however, (e.g. RiLSA 92- German guidelines for the installation of traffic signals [FORSCHUNGSGESELLSCHAFT, 1992], German manual for the dimensioning of road traffic systems [BRILON ET AL, 1994], or even the Highway Capacity Manual 97 [HCM, 1997]) does not provide a method for the evaluation of right turning movements on red at signalised intersections. This constitutes a deficit in particular because exclusive right turning lanes which are controlled by traffic signs 205 or 206 StVO (German road traffic regulations) at signalised intersections are frequently employed in practice. Only for the exception of a shared right turning lane without queuing places, i.e. the 'green arrow' which is subject of many expert discussions since the reunification of Germany, a solution is given by SCHRÄBEL (1997).

The present paper introduces a general solution for the calculation of the capacity benefit by right turning lanes of any selectable length. The plausibility of the solution is demonstrated by simple estimations the results of which can also be used as a simple approximation for dimensioning in practical applications. It could also be proven that the solution by SCHRÄBEL for the green arrow is a special case of the general solution.

2 Prerequisites

Reference is made to Fig. 1 which qualitatively illustrates an intersection with an exclusive right turning lane. Traffic flow 1 is controlled by traffic lights, while flow 2 must give way to flow 3 in accordance to priority rules. Flow 2 can use the turning lane irrespective of the traffic lights until K₁ vehicles from flow 1 have queued during the red time and the K₁th vehicle blocks the lane. The determination of the number K₁ from the intersection geometry is based on the average required queuing length of one vehicle.

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The traffic volumes of the flows of interest are assumed to be \( q_1, q_2 \) and \( q_3 \) [veh/h] or \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) [veh/s], respectively. In the following it is assumed that the time gaps in flow 3 are sufficiently long so that all right turning vehicles (flow 2) which are not hindered by flow 1 can turn during the red stage of flow 1.

\[ F(X_i = K_i) = \frac{(\lambda_i t)^{K_i}}{K_i!} \exp(-\lambda_i t) \quad i = 1, 2 \]  

(1)

3 Approach

3.1 Idea

The solution is based on the following principal considerations: The turning lane is blocked by the vehicles in flow 1 if \( K_1 \) vehicles queue on 'Red' at the stop line. On the assumption of the probability that the event 'turning lane is blocked' occurs at a time \( t^* \) within the time interval of interest (the red time of flow 1), the expected value for the \( X_{21} \) vehicles from flow 2 is calculated which arrive until that time. This expected value multiplied by the previously calculated probability results in the expected value for the value \( X_{22} \) provided the turning lane is blocked exactly at time \( t^* \). Thus, \( X_{22} \) is the number of vehicles from flow 2 which can turn during the red time if the blockage occurs exactly at time \( t^* \). When integrating the expected value over all possible times \( t^* \), the expected value for the number \( X_{23} \) of vehicles is obtained which can pass the turning lane during the red phase if the turning lane is blocked during red. The expected value for the number of vehicles \( X_4 \) must additionally be considered, which can pass on red, if the turning lane is not blocked within this time. The wanted expected value for the number of vehicles \( X \) which can turn on red is obtained as the sum of the expected values for \( X_{23} \) and \( X_{24} \).
3.2 Mathematical Considerations

\[
F(X_1 \geq K_1, t \leq t^*) = 1 - \sum_{K_1=0}^{K_1^{*}} \frac{(\lambda_1 t^*)^{K_1^*}}{K_1^{*!}} \exp(-\lambda_1 t^*)
\]

is the probability that the turning lane is blocked during the time interval \(0 \leq t \leq t^*\). The differential of this probability by the time is the initially wanted probability that the event "turning lane is blocked" occurs at a time \(t^*\) during a unit of time:

\[
f(X_1 \geq K_1, t = t^*) = \frac{d}{dt} F(X_1 \geq K_1, t \leq t^*) =
\]

\[
= \sum_{K_1=0}^{K_1^{*}} \frac{\lambda_1 t^*)^{K_1^*}}{K_1^{*!}} \exp(-\lambda_1 t^*) - \sum_{K_1=1}^{K_1^{*} - 1} \left[\frac{\lambda_1 t^*)^{K_1^{*} - 1}}{(K_1^{*} - 1)!}\right] \exp(-\lambda_1 t^*) =
\]

\[
= \frac{\lambda_1 t^*)^{K_1^{*} - 1}}{(K_1^{*} - 1)!} \exp(-\lambda_1 t^*) \quad \text{for } K_1 \neq 0
\]

The expected value for the vehicles from flow 2 which could arrive and turn until time \(t^*\) at which the turning lane becomes blocked – provided the lane would always block at time \(t^*\) - is obtained from the Poisson distribution:

\[
E(X_{21}) = \sum_{K_1=0}^{\infty} K_1 \frac{(\lambda_2 t^*)^{K_1}}{K_1^{!}} \exp(-\lambda_2 t^*) = \lambda_2 t^*
\]

The expectation \(E(X_{22})\) for the number of vehicles of flow 2 that will be able to pass during the red time if the lane is blocked exactly at time \(t^*\) and the probability for this event (blocking at time \(t^*\)) is considered hence is:

\[
E(X_{22}) = E(X_{21}) \cdot f(X_1 \geq K_1, t = t^*)
\]

The expected value \(E(X_{23})\) results from the integration of \(E(X_{22})\) over all possible times \(t^*\) at which the event \(X_1 \geq K_1\) can occur:

\[
E(X_{23}) = \int_{t=0}^{t^*} E(X_{22}) dt^* = \frac{\lambda_2 K_1}{(K_1 - 1)!} \int_{t=0}^{t^*} \exp(-\lambda_2 t^*) dt^*
\]

The integral of this general solution can be iteratively solved for integer values of \(K_1\) by means of the following formula (No. 450 of the Bronstein Integral Tables):
The probability that the turning lane is not blocked is calculated as:

\[
F(X < K_i, t \leq t^*) = \sum_{K_i=0}^{K_i-1} \frac{(\lambda_i t^*)^{K_i}}{K_i!} \exp(-\lambda_i t^*) = 1 - \int_{t^*}^{t} f(X_i \geq K_i, t = t^*) dt^*
\]  

(8)

From this, the expected value \( E(X_{24}) \) is obtained as:

\[
E(X_{24}) = F(X_i < K_i, t \leq t^*) \cdot E(X_{21}) = \lambda_2 t_r \cdot \sum_{K_i=0}^{K_i-1} \frac{(\lambda_i t^*)^{K_i}}{K_i!} \exp(-\lambda_i t^*) = \\
\lambda_2 t_r \cdot \left( 1 - \int_{t^*}^{t} \frac{\lambda_i t^*^{K_i}}{(K_i-1)!} \exp(-\lambda_i t^*) dt^* \right)
\]  

(9)

The expected value for the number of right turning vehicles which can pass the lane during the red time of flow 1, \( E(X_2) \), is obtained as:

\[
E(X_2) = E(X_{23}) + E(X_{24})
\]  

(10)

Without having been particularly mentioned the equations are only true \( K_i > 0 \). For \( K_i = 0 \) the following derivation formally applies:

\[
F(X_i \geq K_i, t < t^*) = 1 \quad \Rightarrow \quad f(X_i \geq K_i, t = t^*) = 0 \quad \Rightarrow \quad E_i(X_2) = 0
\]  

(11)

\( K = 0 \) means that there is no possibility for turning right during the red stage.

4 Validation and Application

4.1 Calculation Aid

Formula (10) for the general solution contains a recurring term and is therefore of limited use as a "paper and pencil" method. For practical applications it is therefore recommended to use a computation program. The authors offer such a program under the address http://129.187.169.125/~andi/freilauf.html. It can be started by any commercially available browser (e.g. Netscape Communicator 4.61 or Internet Explorer 4.0) and supplies the expected values \( E(X_2) \) for any combination of \( q_1, q_2 \) and \( t_r \) for various queuing lengths \( K_i \).
4.2 Plausibility Check and Approximation Formula

During the red time the maximum number of $X_2^{\text{max}} = t_r \cdot \lambda_2$ vehicles from flow 2 arrive. The expected value $E(X_2)$ must therefore always be less than $X_2^{\text{max}}$. From flow 1 the maximum of $X_1^{\text{max}} = t_r \cdot \lambda_1$ vehicles arrive during the red time. A queuing area that is larger than $X_1^{\text{max}}$ vehicles will therefore hardly increase the benefit.

Under the assumption that the number of vehicles arriving during each cycle invariably causes the turning lane to be blocked during the red time, an upper limit can be given as an approximation for the expected value of the vehicles that can pass:

$$E_{gr}(X_2) = K_1 \frac{\lambda_2}{\lambda_1}$$  \hspace{1cm} (12)

This formula might be used as a simple and rough estimate for the design and dimensioning of a right turning lane at a signalised intersection. An idea of the error that is being made using this approximation is given in the following numerical example.

4.3 Numerical Example

The following hypothetical numerical example is given:

- $t_r = 40$ sec
- $q_1 = 400$ veh/h $\Rightarrow \lambda_1 = 0.1111$ veh/sec
- $q_2 = 300$ veh/h $\Rightarrow \lambda_2 = 0.083$ veh/sec

With these numerical values the following results are obtained for different $K_1$'s:

<table>
<thead>
<tr>
<th>Queuing length</th>
<th>Capacity increase per cycle [veh]</th>
<th>Est'd upper limit [veh]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_1 = 1$</td>
<td>$E(X_2) = 0.741$</td>
<td>$E_{gr}(X_2) = 0.75$</td>
</tr>
<tr>
<td>$K_1 = 2$</td>
<td>$E(X_2) = 1.443$</td>
<td>$E_{gr}(X_2) = 1.5$</td>
</tr>
<tr>
<td>$K_1 = 3$</td>
<td>$E(X_2) = 2.058$</td>
<td>$E_{gr}(X_2) = 2.25$</td>
</tr>
<tr>
<td>$K_1 = 4$</td>
<td>$E(X_2) = 2.544$</td>
<td>$E_{gr}(X_2) = 3.0$</td>
</tr>
<tr>
<td>$K_1 = 5$</td>
<td>$E(X_2) = 2.888$</td>
<td>$E_{gr}(X_2) = 3.75$</td>
</tr>
<tr>
<td>$K_1 = 25$</td>
<td>$E(X_2) = 3.333$</td>
<td>$E_{gr}(X_2) = 18.75$</td>
</tr>
</tbody>
</table>

$K_1 = 1$ represents a green arrow. The capacity increase which can be obtained without structural expenditures at a standard intersection is evident.

$K_1 = 2$ and $K_2 = 3$ represent the usual layout of right turning lanes with a queuing length of one or two, respectively, vehicles. The expected value and savings in waiting time do not increase linearly
with the queuing area but converge for \( K_1 \to \infty \) towards the saturation limit \( E_{\text{max}}(X_2) = 3.333333 \).

When estimating the expected value by means of formula (12) it must, of course, be taken into consideration that due to stochastic fluctuations within one red time of interest fewer vehicles might arrive and, on the other hand, the two traffic flows might merge "unfavourably", so that the "exact" expected value must be less than this estimate. The obtained exact solution is in agreement with these considerations for all calculated \( K_1 \).

### 4.4 Special Case 'Green Arrow'

The green arrow regulation corresponds to \( K_1 = 1 \) of the general solution. This means that once the first vehicle from flow 1 blocks the turning lane, vehicles from flow 2 must wait as well. For this case, the expected value for the number of vehicles from flow 2 which can pass on red according to (10) is:

\[
E(\text{X}_2, K_1 = 1) = \frac{\lambda_2}{\lambda_1} \cdot e^{-\lambda_1 t} \left( -\lambda_1 t_r - 1 + \lambda_2 t_r \right) + \frac{\lambda_2}{\lambda_1} \left( 1 - e^{-\lambda_1 t} \right)
\]

(13)

For a decreasing \( K_1 \) the number of vehicles which must arrive from flow 1 until blockage of the turning lane decreases. With the red time remaining constant the probability of the arrival of \( X_1 \) or more vehicles increases before the lane is blocked. The maximum number of vehicles \( X_2 \) from flow 2 also decreases which can pass the lane before it is blocked; thus the probability increases that exactly this quantity of \( X_2 \) or more vehicles arrive. For small \( K_1 \)’s and reasonable selected, not too short red time, the arrival process can therefore be neglected so only the ratio between flows \( q_1 \) and \( q_2 \) is of significance [COX ET AL., 1971]. This approach for a simplification results in the formula of SCHNABEL (1997):

\[
E(n) = \frac{\lambda_2}{\lambda_1 + \lambda_2} = \frac{\lambda_2}{\lambda_1},
\]

(14)

or in the estimation for \( K_1 = 1 \) given formula (12), respectively.

In the general approach the arrival process of the vehicles is neglected if \( t \) increases because the arrival probability of more than \( X_1 \) vehicles and more than \( X_2 \) vehicles for any \( X_1 \) and \( X_2 \) converges towards 1 for an infinitely long red time. Going to the limits of \( t \) in formula (13) yields:

\[
\lim_{t_r \to \infty} E(\text{X}_2, K_1 = 1) = \frac{\lambda_2}{\lambda_1}
\]

(15)

The approximation solution according to SCHNABEL (1997) is therefore included as a special case in the general solution.
5 Summary

A closed stochastic and analytical approach for a quantitative evaluation of right turns on red at signalised intersections has been presented for any length of the turning lane. An expected value for the number of vehicles which can turn on red can be calculated by means of an analytical equation without the computation of dynamic models. A calibration of model parameters is not required. The general solution also covers the green arrow; in addition, the approximation solution according to SCHNABEL (1997) is included in the general solution as a special case.

An exact planning tool can be generated by a user-friendly implementation of the formulas. This was demonstrated by means of the Javaapplet which is available under http://129.187.169.125/~andi/freilauf.html. The formulas also lend themselves for a conversion in a freely programmable pocket calculator.

The presented approach can be used as a basis for further amendments of the analytical model. It is intended for future studies to take into account the influence of the traffic volume of the priority flow and the exact calculation of the mean savings in waiting time.

6 References


FORSCHUNGSGESELLSCHAFT FÜR STRASSEN- UND VERKEHRSWESEN (1992): Richtlinien für Lichtsignalanlagen - RiLSA
